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Summary

The paraxial trajectories in electron lenses are derived from an integral equation. The Liouville-Neumann expansions of the solutions of this equation lead to expressions for the magnification, the focal distances and the positions of the focal points and cardinal points. The numbers of integrations to be performed in the individual terms of the expansions used to describe lens characteristics are reduced to half, as compared with the normal treatment. The focal and cardinal points are defined as osculating elements similar to those introduced by G l a s e r.

§ 1. *Introduction.* The paraxial theory of rotationally symmetric electron lenses is based on the neglect of second and higher powers of the distance ϱ from the axis when deriving the equation of motion of the electrons. The linear differential equation thus obtained is of the following form for electrons originally in a meridional plane:

$$\frac{d^2y}{dz^2} + k^2(z)\, y = 0. \tag{1}$$

The coordinate z refers to distances in the direction of the symmetry-axis. In the most general case of a combined electrostatic and magnetic lens the dependent variable is given by $y = \varrho(z) V^{\frac{1}{4}}(0, z)$, $V(\varrho, z)$ being the potential in e.s.u. of the electric field. The coefficient $k^2(z)$ depends on the electrostatic potential $V(z)$ and the magnetic field $H(z)$ along the axis according to [1])

$$k^2(z) = \frac{3}{16}\frac{V'^2}{V^2} + \frac{e}{8mc^2}\frac{H^2}{V}. \tag{2}$$

The effect of space charge is neglected here.

Only in a few exceptional cases is $k^2(z)$ of such a form that (1) can be solved analytically for a field approximately realized in practice. In this connection we mention in particular the case

$$k^2(z) = \frac{k_0^2}{(1 + z^2/a^2)^2}$$

that corresponds to the bell shaped magnetic field

$$H(z) = \frac{H_0}{1 + z^2/a^2},$$

which has been investigated thoroughly by W. Glaser[2]). This field leads to expressions for $\varrho(z)$ that depend on trigonometric functions, viz.

$$\varrho = A\left(1 + \frac{z^2}{a^2}\right)^{\frac{1}{2}} \sin\left\{(1 + k_0^2 a^2)^{\frac{1}{2}} \left(\arctan \frac{z}{a} - \varphi_0\right)\right\}.$$

Further, a magnetic field corresponding to electron trajectories described by Legendre functions is given by

$$H(z) = \frac{H_0}{\cosh(z/a)}.$$

The most general solution in this case, viz.

$$\varrho = AP_\nu\left\{\tanh\left(\frac{z}{a}\right)\right\} + BQ_\nu\left\{\tanh\left(\frac{z}{a}\right)\right\},$$

depends on Legendre functions, the degree of which follows from the relation

$$\nu(\nu + 1) = \frac{ea^2}{8mc^2}\,\frac{H_0^2}{V}.$$

As a rule, however, it is necessary to resort to some numerical method or other in order to derive results from (1) when the field distribution along the axis is given by experimental data. In particular, approximation methods introducing finite differences instead of differential quotients have been frequently applied. A special method based on successive approximating integrations has been worked out by M. v. Ments and J. B. Le Poole[3]). In the present paper the method used by these authors is simplified by lowering the number of integrations occurring in the various correction terms. The treatment is based on an integral equation discussed in the next sections.

§ 2. *Derivation of the paraxial integral equation.* Let us consider the general case of an immersion lens. The value $k^2(z_0)$ of $k^2(z)$ at the object plane $z = z_0$ may thus differ from zero. The equation to be derived will refer to a special electron trajectory, characterized by the values y_0 and y'_0 of y and dy/dz respectively at the object plane. According to the relation $y = \varrho V^{\frac{1}{4}}$ these quantities are proportional to the distance ϱ_0 from the axis and to the tangent of the angle with the z-axis at the object point from which the electron is starting. A first integration of the paraxial equation, viz. $d^2y/dz^2 = - k^2(z)\, y$, then yields

$$\frac{dy}{dz} = y'_0 - \int_{z_0}^{z} d\zeta\, k^2(\zeta)\, y(\zeta).$$

A second integration gives

$$y = y_0 + y'_0\,(z - z_0) - \int_{z_0}^{z} d\zeta_1 \int_{z_0}^{\zeta_1} d\zeta\, k^2(\zeta)\, y(\zeta).$$

By inverting the order of integration the double integral proves to be equal to

$$\int_{z_0}^{z} d\zeta\, k^2(\zeta)\, y(\zeta) \int_{\zeta}^{z} d\zeta_1 = \int_{z_0}^{z} d\zeta\, k^2(\zeta)\, y(\zeta)\, (z - \zeta).$$

It is to be noted that this reduction to a single integral halves the numbers of integrations occurring in the expansions to be derived in the following sections.

We thus arrive at the following integral equation:

$$y(z) = y_0 + y'_0(z - z_0) - \int_{z_0}^{z} d\zeta\, k^2(\zeta)\, (z - \zeta)\, y(\zeta), \tag{3}$$

which completely determines the paraxial behaviour of the lens system.

§ 3. *Solution of the integral equation by the Liouville-Neumann method.* The first step in this method [4]) consists in substituting into (3) for $y(\zeta)$ its value according to the equation itself, that is

$$y(\zeta) = y_0 + y'_0\,(\zeta - z_0) - \int_{z_0}^{\zeta} d\zeta_1\, k^2(\zeta_1)\, (\zeta - \zeta_1)\, y(\zeta_1).$$

The evaluation of this first substitution yields

$$y(z) = y_0 + y_0'(z - z_0) - \int_{z_0}^{z} d\zeta\, k^2(\zeta)\,(z - \zeta)\,\{y_0 + y_0'(\zeta - z_0)\} + \\ + \int_{z_0}^{z} d\zeta\, k^2(\zeta)\,(z - \zeta) \int_{z_0}^{\zeta} d\zeta_1\, k^2(\zeta_1)\,(\zeta - \zeta_1)\, y(\zeta_1). \quad (4)$$

The second step amounts to substituting for $y(\zeta_2)$ in the last integral once again its value according to (3), that is

$$y(\zeta_1) = y_0 + y_0'(\zeta_1 - z_0) - \int_{z_0}^{\zeta_1} d\zeta_2\, k^2(\zeta_2)\,(\zeta_1 - \zeta_2)\, y(\zeta_2).$$

The third step consists of a further application of (3) to $y(\zeta_2)$, and so on. By repeating this procedure ad infinitum we obtain the series

$$y(z) = y_0 + y_0'(z - z_0) - \int_{z_0}^{z} d\zeta\, k^2(\zeta)\,(z - \zeta)\,\{y_0 + y_0'(\zeta - z_0)\} + \\ + \int_{z_0}^{z} d\zeta\, k^2(\zeta)\,(z - \zeta) \int_{z_0}^{\zeta} d\zeta_1\, k^2(\zeta_1)\,(\zeta - \zeta_1)\,\{y_0 + y_0'(\zeta_1 - z_0)\} - \\ - \int_{z_0}^{z} d\zeta\, k^2(\zeta)\,(z - \zeta) \int_{z_0}^{\zeta} d\zeta_1\, k^2(\zeta_1)\,(\zeta - \zeta_1) \int_{z_0}^{\zeta_1} d\zeta_2\, k^2(\zeta_2)\,(\zeta_1 - \zeta_2)\,\cdot \\ \cdot \{y_0 + y_0'(\zeta_2 - z_0)\} + \ldots . \quad (5)$$

This expansion with the aid of so-called *iterated* kernels represents the exact solution of (3) in the case of convergence. A sufficient condition for convergence is that the product of the integration interval and the maximum value of the main kernel should not exceed unity. We are only interested in the space bounded by the object plane $z = z_0$ and the corresponding image plane $z = z_i$. The main kernel being given by $k^2(\zeta)\,(z - \zeta)$, with a maximum value $\max k^2(z) \times (z_i - z_0)$, the above sufficient condition for convergence amounts to

$$\max k^2(z) \times (z_i - z_0)^2 < 1.$$

For a magnetic lens this reduces to

$$H^2_{max}\,(z_i - z_0)^2 < \frac{8mc^2}{e} V, \quad (6)$$

or

$$H^2_{max}\,(z_i - z_0)^2 < 45.2\ V$$

if H_{max}, $z_i - z_0$ and V are expressed in gauss, centimetre and volt respectively. The condition (6) can be given in a more illustrative form by replacing the magnetic field H by the curvature

$$R = \frac{mvc}{eH}$$

of the trajectory of an electron moving with velocity v perpendicularly to a homogeneous field of the same strength. The equality (6) then reads

$$z_i - z_0 < 2\, R_{max}.$$

The expansions to be discussed in the next sections therefore converge more rapidly according as the quantity $(z_i - z_0)/2\, R_{max}$ is smaller.

§ 4. *Particular solutions. Determination of the image plane.* The expansion (5) disintegrates automatically into

$$y(z) = y_0\, y_1(z) + y_0'\, y_2(z), \tag{7}$$

in which occur two particular solutions $y_1(z)$ and $y_2(z)$ of the original equation (1). These special solutions are given by

$$y_1(z) = 1 - \int_{z_0}^{z} \mathrm{d}\zeta\, k^2(\zeta)\,(z-\zeta) + \int_{z_0}^{z} \mathrm{d}\zeta\, k^2(\zeta)\,(z-\zeta) \int_{z_0}^{\zeta} \mathrm{d}\zeta_1\, k^2(\zeta_1)(\zeta-\zeta_1) -$$
$$- \int_{z_0}^{z} \mathrm{d}\zeta\, k^2(\zeta)\,(z-\zeta) \int_{z_0}^{\zeta} \mathrm{d}\zeta_1\, k^2(\zeta_1)\,(\zeta-\zeta_1) \int_{z_0}^{\zeta_1} \mathrm{d}\zeta_2\, k^2(\zeta_2)\,(\zeta_1-\zeta_2) + \dots, \tag{8}$$

and

$$y_2(z) = z - z_0 -$$
$$- \int_{z_0}^{z} \mathrm{d}\zeta\, k^2(\zeta)(z-\zeta)(\zeta-z_0) + \int_{z_0}^{z} \mathrm{d}\zeta\, k^2(\zeta)(z-\zeta) \int_{z_0}^{\zeta} \mathrm{d}\zeta_1 k^2(\zeta_1)(\zeta-\zeta_1)(\zeta_1-z_0) -$$
$$- \int_{z_0}^{z} \mathrm{d}\zeta\, k^2(\zeta)(z-\zeta) \int_{z_0}^{\zeta} \mathrm{d}\zeta_1\, k^2(\zeta_1)(\zeta-\zeta_1) \int_{z_0}^{\zeta_1} \mathrm{d}\zeta_2 k^2(\zeta_2)(\zeta_1-\zeta_2)(\zeta_2-z_0) + \dots. \tag{9}$$

The solution $y_1(z) = V^{\frac{1}{4}}(z)\ \varrho_1(z)$ is characterized by $y_1(z_0) = 1$ and $y_1'(z_0) = 0$. Therefore $y_1(z)$ represents the trajectory of the electrons leaving the object at a point, a distance $\varrho_1(z_0) = 1/V^{\frac{1}{4}}\,(z_0)$ away from the axis in a direction to be determined from $y_1'(z_0) = 0$; for magnetic lenses (V independent of z) this direction is parallel to the axis. The function $y_2(z) = V^{\frac{1}{4}}(z)\ \varrho_2(z)$ likewise describes the electrons leaving the axis point of the object ($y_2(z_0) = 0$) in a direction determined by $y_2'(z_0) = 1$ (see fig. 1).

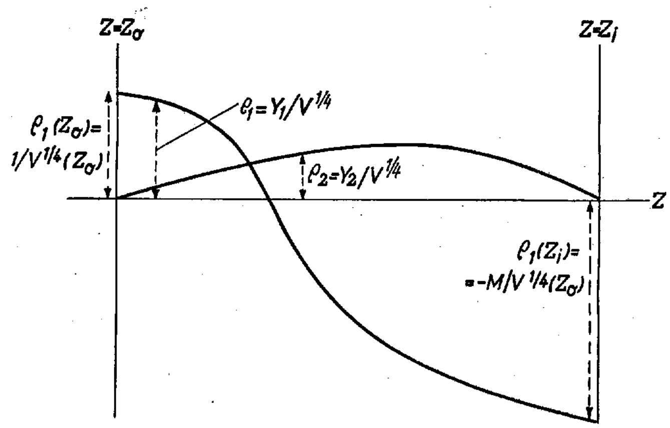

Fig. 1. Particular solutions of the integral equation.

The situation of the image plane $z = z_i$ now follows from the condition that any trajectory starting from a fixed point of the object plane (value of $\varrho(z_0) = y(z_0)/V^{\frac{1}{4}}(z_0)$ given) should traverse the image plane at the same point. The value of $y(z_i)$ must therefore be independent of y_0', which results in the condition $y_2(z_i) = 0$. By equating to zero the left-hand member of (9) we thus arrive at a relation for the unknown $z = z_i$ which is simply a linear equation if we leave out of consideration the occurrence of z in the upper limits of integration. In this way we obtain

$$z_i = \frac{z_0 - \int_{z_0}^{z_i} \mathrm{d}\zeta\, k^2(\zeta)\zeta(\zeta - z_0) + \int_{z_0}^{z_i} \mathrm{d}\zeta\, k^2(\zeta)\zeta \int_{z_0}^{\zeta} \mathrm{d}\zeta_1\, k^2(\zeta_1)(\zeta - \zeta_1)(\zeta_1 - z_0) - \dots}{1 - \int_{z_0}^{z_i} \mathrm{d}\zeta\, k^2(\zeta)(\zeta - z_0) + \int_{z_0}^{z_i} \mathrm{d}\zeta\, k^2(\zeta) \int_{z_0}^{\zeta} \mathrm{d}\zeta_1\, k^2(\zeta_1)(\zeta - \zeta_1)(\zeta_1 - z_0) - \dots}, \quad (10)$$

which relation does not determine the image plane explicitly owing to the dependence of the integration limits on z_i. In practice, however, even in the case of an immersion lens the field is often already negligible at the image plane, so that the limits $\zeta = z_i$ may just as well be replaced by a quantity independent of z_i (for instance by that corresponding to the edge of the field or by $\zeta = \infty$).

§ 5. *The magnification.* The magnification M can be derived from the solution $y_1(z)$ according to the formula:

$$M = -\frac{\varrho_1(z_i)}{\varrho_1(z_0)} = -V^{\frac{1}{2}}(z_0)\,\varrho_1(z_i) = -\frac{V^{\frac{1}{2}}(z_0)}{V^{\frac{1}{2}}(z_i)}\,y_1(z_i). \qquad (11)$$

The minus sign indicates that M represents a positive quantity if the electrons leaving an off-axis object point in a meridional plane do intersect the axis once (or occasionally an odd number of times) before arriving at the image point (see fig. 1).

We could derive an expression for $y_1(z_i)$ and thus for M simply by substituting $z = z_i$ in (8). The resulting formula, however, would still depend on z_i, which occurs both in the integrand and in the integration limits of the various terms. A more useful formula without having z_i in the integrands can be deduced from $y_2(z)$ by taking into account the Wronski property relative to y_1 and y_2, viz.

$$y_1(z)\,y_2'(z) - y_2(z)\,y_1'(z) = \text{constant}.$$

The value of the constant follows from the substitution $z = z_0$, remembering that $y_1(z_0) = 1$, $y_2(z_0) = 0$ and $y_2'(z_0) = 1$. Hence

$$y_1(z)\,y_2'(z) - y_2(z)\,y_1'(z) = 1. \qquad (12)$$

A further application of (12) at $z = z_i$ yields, in view of (11) and $y_2(z_i) = 0$,

$$\frac{1}{M} = -\frac{V^{\frac{1}{2}}(z_i)}{V^{\frac{1}{2}}(z_0)}\,y_2'(z_i). \qquad (13)$$

A series expansion for $y_2'(z_i)$ is easily derived from (9). The resulting final formula for $1/M$ becomes

$$\frac{1}{M} = \frac{V^{\frac{1}{2}}(z_i)}{V^{\frac{1}{2}}(z_0)}\Big\{-1 + \int_{z_0}^{z_i} \mathrm{d}\zeta\, k^2(\zeta)\,(\zeta - z_0) - \\ - \int_{z_0}^{z_i} \mathrm{d}\zeta\, k^2(\zeta) \int_{z_0}^{\zeta} \mathrm{d}\zeta_1\, k^2(\zeta_1)\,(\zeta - \zeta_1)\,(\zeta_1 - z_0) + \\ + \int_{z_0}^{z_i} \mathrm{d}\zeta\, k^2(\zeta) \int_{z_0}^{\zeta} \mathrm{d}\zeta_1\, k^2(\zeta_1)\,(\zeta - \zeta_1) \int_{z_0}^{\zeta_1} \mathrm{d}\zeta_2\, k^2(\zeta_2)(\zeta_1 - \zeta_2)(\zeta_2 - z_0) - \ldots\Big\}. \qquad (14)$$

The magnification can be computed from this series without knowing the exact situation of the image plane provided the field is negligible there. In fact, in that case the upper limit of integration z_i can be replaced once again by ∞ or by the z-coordinate of the edge of the field.

The expansion (13) can be transformed into an other one, the terms of which show a more symmetrical structure. The new expansion is obtained by first shifting the factor of the integrand which contains z_0 to the front by means of a number of reversals of orders of integration. By additional reversals it proves to be possible to obtain integrals the upper limit of which equals z_i. The final expansion then reads

$$\frac{1}{M} = \frac{V^{\frac{1}{2}}(z_i)}{V^{\frac{1}{2}}(z_0)} \Big\{ -1 + \int_{z_0}^{z_i} d\zeta_1\, k^2(\zeta_1)\,(\zeta_1 - z_0) - \int_{z_0}^{z_i} d\zeta_1\, k^2(\zeta_1)\,(\zeta_1 - z_0) \int_{\zeta_1}^{z_i} d\zeta_2\, k^2(\zeta_2)\,(\zeta_2 - \zeta_1) + \int_{z_0}^{z_i} d\zeta_1\, k^2(\zeta_1)(\zeta_1 - z_0) \int_{\zeta_1}^{z_i} d\zeta_2\, k^2(\zeta_2)\,(\zeta_2 - \zeta_1) \int_{\zeta_2}^{z_i} d\zeta_3\, k^2(\zeta_3)(\zeta_3 - \zeta_2) - \dots \Big\}. \quad (15)$$

We also give the corresponding series in practical units for magnetic lenses. For these lenses we have, according to (2),

$$k^2(z) = \frac{0.0221}{V} H^2(z) \quad (16)$$

if H and V are expressed in gauss and volt respectively. The first few terms of (15) thus yield for the magnification by magnetic lenses

$$\frac{1}{M} = -1 + \frac{0.0221}{V} \int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1)\,(\zeta_1 - z_0) - \frac{0.000488}{V^2} \int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1)\,(\zeta_1 - z_0) \int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)\,(\zeta_2 - \zeta_1) + \frac{1{,}079 . 10^{-5}}{V^3} \cdot \int_{z_0}^{z} d\zeta_1\, H^2(\zeta_1)\,(\zeta_1 - z_0) \int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)\,(\zeta_2 - \zeta_1) \int_{\zeta_2}^{z_i} d\zeta_3\, H^2(\zeta_3)(\zeta_3 - \zeta_2) - \dots . \quad (17)$$

§ 6. *The dependence of the osculating constants on the magnification.* The function $k^2(z)$ of (2) decreases gradually on both sides of its maximum without being zero at the object and image planes. Therefore any electron lens constitutes, strictly speaking, an immersion lens. As a consequence it is impossible to define focal and cardinal points which are independent of the special position of these planes, though the dependence is negligible provided $k^2(z)$ is very small at $z = z_0$ and $z = z_i$. Glaser [5]) introduced so-called osculating focal and cardinal points which take into account the influence of $k^2(z_0)$ and $k^2(z_i)$. His definitions are chosen such that the tangents at the intersections of the trajectories with $z = z_0$ and $z = z_i$ show the same properties, in connection with the osculating elements, as the rectilinear trajectories in ordinary optics. A disadvantage of Glaser's definitions is that they do not apply to all electron lenses. In what follows we start from slightly different definitions which are applicable throughout and which are identical with Glaser's definitions in so far as the latter exist. Our definitions correspond, e.g., to that of the osculating elements used in planetary mechanics; in the latter these elements determine the ellipse which approximates best the actual trajectory in the vicinity of the point under consideration.

The four osculating elements to be introduced here are the focal points F_0 and F_i and the cardinal points H_0 and H_i in object and

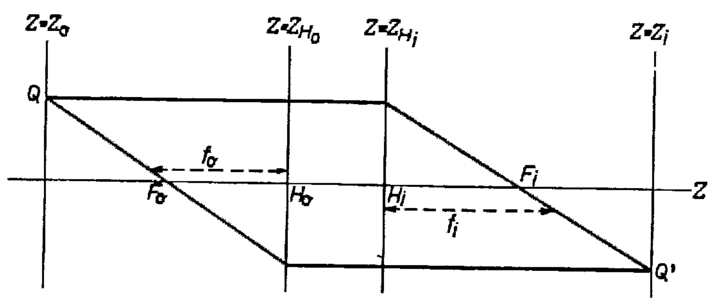

Fig. 2. Showing the significance of the osculating elements.

image space respectively; the focal distances $f_0 = z_{H_0} - z_{F_0}$ and $f_i = z_{F_i} - z_{H_i}$ will be defined as positive if the situation corresponds to that of fig. 2. The osculating elements are to be chosen in accordance with the concepts of elementary geometrical lens theory, that is, such that the image Q' of the object point Q may be found as if the electron trajectories were straight lines. The corresponding situation as shown in fig. 2 recalls the well-known relations

$$z_{H_0} - z_{F_0} = M(z_{F_0} - z_0); \qquad z_i - z_{Fi} = M(z_{Fi} - z_{Hi}). \tag{18}$$

In these equations z_0, z_i and M are the only known quantities in our case. For a determination of the four unknown coordinates z_{F_0}, z_{Fi}, z_{H_0} and z_{Hi} fixing the osculating elements we need two further relations. The latter follow from the condition that the *same* osculating elements should also apply to a second position of the object plane (and the corresponding image plane) that is infinitely close to the original one. This implies that (18) can be differentiated with respect to z_0 while keeping z_{H_0}, z_{Hi}, z_{F_0} and z_{F_i} constant. In this way we get the additional relations

$$0 = (z_{F_0} - z_0)\frac{\mathrm{d}M}{\mathrm{d}z_0} - M; \qquad \frac{\mathrm{d}z_i}{\mathrm{d}z_0} = (z_{Fi} - z_{Hi})\frac{\mathrm{d}M}{\mathrm{d}z_0}. \tag{19}$$

By solving the four unknowns from the four equations (18) and (19) we obtain

$$\begin{aligned} z_{F_0} &= z_0 + \frac{M}{\mathrm{d}M/\mathrm{d}z_0}; & z_{Fi} &= z_i - \frac{M}{\mathrm{d}M/\mathrm{d}z_0}\frac{\mathrm{d}z_i}{\mathrm{d}z_0}, \\ z_{H_0} &= z_0 + \frac{M + M^2}{\mathrm{d}M/\mathrm{d}z_0}; & z_{Hi} &= z_i - \frac{1 + M}{\mathrm{d}M/\mathrm{d}z_0}\frac{\mathrm{d}z_i}{\mathrm{d}z_0}. \end{aligned} \tag{20}$$

Further, the differential coefficient $\mathrm{d}z_i/\mathrm{d}z_0$ can be connected with M as follows. We consider an object plane $z = z_0 + \Delta z_0$ and a corresponding image plane $z = z_i + \Delta z_i$, differing from the original ones (for which $\Delta z_0 = \Delta z_i = 0$). The quotient of any two particular solutions of (1) assumes identical values at $z = z_0 + \Delta z_0$ and $z = z_i + \Delta z_i$, a property which expresses the homogeneous character of the magnication. An application of this property to the two special solutions y_1 and y_2 (see fig. 1) yields

$$y_1(z_0 + \Delta z_0)\, y_2(z_i + \Delta z_i) - y_2(z_0 + \Delta z_0)\, y_1(z_i + \Delta z_i) = 0.$$

Next we obtain from a differentiation with respect to Δz_0

$$\frac{\mathrm{d}\Delta z_i}{\mathrm{d}\Delta z_0} = -\frac{y_1'(z_0 + \Delta z_0)\, y_2(z_i + \Delta z_i) - y_2'(z_0 + \Delta z_0)\, y_1(z_i + \Delta z_i)}{y_1(z_0 + \Delta z_0)\, y_2'(z_i + \Delta z_i) - y_2(z_0 + \Delta z_0)\, y_1'(z_i + \Delta z_i)}.$$

We can pass to the limit for $\Delta z_0 \to 0$, bearing in mind that $y_1(z_0) = y_2'(z_0) = 1$ and $y_2(z_0) = y_2(z_i) = 0$. Hence $\mathrm{d}z_i/\mathrm{d}z_0 = y_1(z_i)/y_2'(z_i)$, or, in view of (11) and (13),

$$\frac{dz_i}{dz_0} = M^2 \left\{ \frac{V(z_i)}{V(z_0)} \right\}^{\frac{1}{2}}. \tag{21}$$

A substitution into (20) leads to the following formulae if we introduce the focal distances $f_0 = z_{H_0} - z_{F_0}$ and $f_i = z_{F_i} - z_{H_i}$ instead of z_{H_0} and z_{H_i}:

$$z_{F_0} = z_0 + \frac{M}{dM/dz_0}; \qquad z_{F_i} = z_i - \left\{\frac{V(z_i)}{V(z_0)}\right\}^{\frac{1}{2}} \frac{M^3}{dM/dz_0},$$
$$f_0 = \frac{M^2}{dM/dz_0}; \qquad f_i = \left\{\frac{V(z_i)}{V(z_0)}\right\}^{\frac{1}{2}} \frac{M^2}{dM/dz_0}. \tag{22}$$

These final expressions prove to be identical with those derived by Glaser. For a comparison with the formulae of this author, however, it is necessary to link $M = -\varrho_1(z_i)$ to those particular solutions of (1) that correspond to electrons arriving at either side from infinity in the direction of the z-axis.

§ 7. *Explicit formulae for the osculating constants of magnetic lenses.* Expansions similar to (15) can now be derived from (22) for the osculating elements. The evaluation can be based on (15) and its derivative with respect to z_0. In the case of pure magnetic lenses we have $V(z_0) = V(z_i)$, so that the focal distances $f_0 = f_i = M^2/(dM/dz_0)$ can be derived from

$$\frac{1}{f} = -\frac{d(1/M)}{dz_0}.$$

From a differentiation of (17) we obtain the following expansion in practical units:

$$\frac{1}{f} = \frac{0.0221}{V}\int_{z_0}^{z_i} d\zeta\, H^2(\zeta) - \frac{0.000488}{V^2}\int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1)\int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)(\zeta_2-\zeta_1) +$$
$$+ \frac{1.079.10^{-5}}{V^3}\int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1)\int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)(\zeta_2-\zeta_1)\int_{\zeta_2}^{z_i} d\zeta_3\, H^2(\zeta_3)(\zeta_3-\zeta_2) - \dots$$
$$- H^2(z_i)\, M^2 \left\{ \frac{0.0221}{V}(z_i - z_0) - \frac{0.000488}{V^2}\int_{z_0}^{z_i} d\zeta\, H^2(\zeta)\,(\zeta - z_0)\,(z_i - \zeta) + \right.$$
$$\left. + \frac{1.079.10^{-5}}{V^3}\int_{z_0}^{z_i} d\zeta_1 H^2(\zeta_1)(\zeta_1 - z_0)\int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)(\zeta_2-\zeta_1)(z_i-\zeta_2) - \dots \right\}. \tag{21}$$

The second group of terms can be neglected if the field at the image plane is very small. Otherwise we may consider these terms as correction terms which can be taken into account by substituting approximate values for M and z_i. We emphasize that the number of integrations occurring in the terms of the first group is lower than the corresponding number in the conventional formulae [6].

The positions of the focal and cardinal points can now be derived from (17) and (21) with the aid of the following relations, which result from (18):

$$z_{F_0} = z_0 + \frac{f}{M} = z_0 + \frac{1/M}{1/f}; \qquad z_{H_0} = z_0 + \frac{1 + 1/M}{1/f}.$$

The corresponding formulae for the elements of the image space are always found by interchanging the role of z_0 and z_i, which implies that M has to be replaced by $1/M$. The explicit form of these formulae is much simplified if the additional terms arising from the field at the image (or object plane) may be neglected. In that case we obtain, e.g., the following expression for the position of the cardinal point in the object space:

$$z_{H_0} = \frac{N}{D},$$

in which

$$N = \int_{z_0}^{z_i} d\zeta\, H^2(\zeta)\, \zeta - \frac{0.0221}{V} \int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1)\, \zeta_1 \int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)\, (\zeta_2 - \zeta_1) +$$

$$+ \frac{0.000488}{V^2} \int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1)\, \zeta_1 \int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)(\zeta_2 - \zeta_1) \int_{\zeta_2}^{z_i} d\zeta_3 H^2(\zeta_3)(\zeta_3 - \zeta_2) - \ldots,$$

and

$$D = \int_{z_0}^{z_i} d\zeta\, H^2(\zeta) - \frac{0.0221}{V} \int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1) \int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)\, (\zeta_2 - \zeta_1) +$$

$$+ \frac{0.000488}{V^2} \int_{z_0}^{z_i} d\zeta_1\, H^2(\zeta_1) \int_{\zeta_1}^{z_i} d\zeta_2\, H^2(\zeta_2)\, (\zeta_2 - \zeta_1) \int_{\zeta_2}^{z_i} d\zeta_3 H^2(\zeta_3)(\zeta_3 - \zeta_2) \ldots.$$

For very fast electrons or very weak fields the first term of the numerator and that of the denominator yield a sufficiently accurate approximation, viz.

$$z_{H_0} \sim \frac{\int_{z_0}^{z_i} d\zeta \, H^2(\zeta)\, \zeta}{\int_{z_0}^{z_i} d\zeta \, H^2(\zeta)} .$$

In this approximation both cardinal points have the same z-coordinate as the centre of gravity of the part of the curve representing H^2 as a function of z that is bounded by the curve, the z-axis and the ordinates $z = z_0$ and $z = z_i$. It is striking that in such expressions H^2 and not H constitutes the dominating quantity.

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REFERENCES

1) Zworykin, V. K., et al., Electron Optics and the Electron Microscope, p. 505; New York, 1948.
2) Glaser, W., Z. Phys. **117** (1941) 285.
3) Ments, M. van, and J. B. Le Poole, Appl. sci. Res. **B1** (1947) 3.
4) Whittaker, E. T., and G. N. Watson, A course of modern analysis p. 221; Cambridge, 1940.
5) Glaser, W., Z. angew. Math. Phys. **1** (1950) 363.
6) See Cosslett, Introduction to Electron Optics; Oxford, 1946, formula IV, 64.